J. DIFFERENTIAL GEOMETRY
37 (1993) 505-522

ON THE MODULI SPACE OF VECTOR BUNDLES ON THE FIBERS OF THE UNIVERSAL CURVE

ALEXIS KOUVIDAKIS

Dedicated to the memory of S. K. Pichorides

Abstract

In this paper we describe the Picard group of the variety $\mathscr{U}(r, d)$ which parametrizes semistable vector bundles of rank r and degree d on the fibers of the universal curve $\mathscr{C}_g$. The bundle $\mathscr{U}(r, d)$ lies over the moduli space $\mathscr{M}_g^0$ of smooth curves of genus g $(g \geq 3)$ without automorphisms.

1. Introduction

We denote by $\mathscr{M}_g^0$ the moduli space of smooth curves of genus g $(g \geq 3)$ without automorphisms. To this space we can associate various varieties: The universal curve $\pi: \mathscr{C}_g \to \mathscr{M}_g^0$ which is a bundle with fiber the curve C over the point $[C] \in \mathscr{M}_g^0$; the variety $q: \mathscr{U}(r, d) \to \mathscr{M}_g^0$ with fiber over $[C]$ the space $U_C(r, d)$, which parametrizes semistable vector bundles of rank r and degree d on C—for the definition see [9]. In the special case when $r = 1$, this becomes the Jacobian variety $p: \mathscr{J}^d \to \mathscr{M}_g^0$ of degree d with fiber $J^d(C)$ over the point $[C]$, which parametrizes line bundles of degree d on C.

The Picard groups of $\mathscr{M}_g^0$ and $\mathscr{C}_g$ have been described by Harer, Arbarello and Cornalba (see [1]). The $\operatorname{Pic}\mathscr{M}_g^0$ is generated by the determinant λ of the Hodge bundle. On the other hand, the restriction of a line bundle on $\mathscr{C}_g$ to the fibers of π is something "canonical", namely a multiple of the canonical bundle (Franchetta's problem, see [1]). Therefore the relative Picard group $\operatorname{Pic}(\mathscr{C}_g/\mathscr{M}_g^0)$ is generated by the relative dualizing sheaf ω_π of the family π and the $\operatorname{Pic}\mathscr{C}_g$ is the free abelian group with generators ω_π and $\pi^*\lambda$.

In this paper we prove that a similar phenomenon holds for line bundles on $\mathscr{U}(r, d)$. The restriction of a line bundle on $\mathscr{U}(r, d)$ to a fiber

Received January 13, 1992.

$U_C(r, d)$ is again something "canonical" in the sense that we explain in §3. Before we continue, let us note that we have a natural isomorphism $\mathscr{U}(r, d) \cong \mathscr{U}(r, d+r(2g-2))$ given by $E \mapsto E \otimes K$, where K the canonical bundle. Using this, it is enough to describe the $\operatorname{Pic}\mathscr{U}(r, d)$ for large values of the degree d.

2. Some properties of θ divisors

We state here some technical lemmas concerning properties of θ divisors on the Jacobian of a smooth curve. First a notation. By fixing a line bundle $L \in J^{-d+g-1}(C)$, the locus of $\{M \in J^d(C)$ such that $h^0(M \otimes L) \geq 1\}$ is of codimension one in $J^d(C)$. We denote by θ_L the line bundle on $J^d(C)$ corresponding to this divisor (or sometimes the divisor itself).

Lemma 1. *Let $\mathscr{A}$ be an abelian variety, and $\mathscr{L}$ a principal polarization on $\mathscr{A}$. Then the map $\phi_{\mathscr{L}}: \mathscr{A} \to \operatorname{Pic}(\mathscr{A})$ which sends $A \mapsto T_A^*\mathscr{L} \otimes \mathscr{L}^{-1}$ is a group homomorphism.*

Proof. See [8, p. 59, Corollary 4].

Lemma 2. *$J^0(C)$ is naturally isomorphic to the variety $\operatorname{Pic}^0 J^d(C)$ which parametrizes the line bundle of class 0 on $J^d(C)$.*

Proof. Fix a principal polarization θ_M on $J^d(C)$, where $M \in J^{-d+g-1}(C)$ following the above notation. Consider the map $J^0(C) \to \operatorname{Pic}^0 J^d(C)$ which sends $L \mapsto \theta_{M\otimes L} \otimes \theta_M^{-1}$. As it turns out this does not depend on the choice of M and is an isomorphism (see [8] for details).

Lemma 3. *If $A, B \in J^{-d+g-1}(C)$ with $A^n = B^n$, then $\theta_A^n = \theta_B^n$ on $J^d(C)$. More generally, if $A_i, B_j \in J^{-d+g-1}(C)$ with $\bigotimes_{i=1}^s A_i^{n_i} = \otimes_{j=1}^t B_j^{m_j}$, $\sum_{i=1}^s n_i = \sum_{j=1}^t m_j$, then $\bigotimes_{i=1}^s \theta_{A_i}^{n_i} = \bigotimes_{j=1}^t \theta_{B_j}^{m_j}$.*

Proof. We prove the general case. It is enough to prove the lemma for the case $d = 0$. Then using an identification of $J^d(C)$ with $J^0(C)$ it is true for all d. Fix a polarization θ_C on $J^d(C)$. Then $\theta_{A_i} = T^*_{A_i\otimes C^{-1}}\theta_C$. We want to prove that

$$\bigotimes_{i=1}^{s} \theta_{A_i}^{n_i} = \bigotimes_{j=1}^{t} \theta_{B_j}^{m_j},$$

or

$$\bigotimes_{i=1}^{s} (\theta_{A_i}^{n_i} \otimes \theta_C^{-n_i}) = \bigotimes_{j=1}^{t} (\theta_{B_j}^{m_j} \otimes \theta_C^{-m_j}),$$

or

$$\bigotimes_{i=1}^{s}(T^*_{A_i\otimes C^{-1}}\theta_C^{n_i}\otimes\theta_C^{-n_i})=\bigotimes_{j=1}^{t}(T^*_{B_j\otimes C^{-1}}\theta_C^{m_j}\otimes\theta_C^{-m_j}),$$

or

$$\bigotimes_{i=1}^{s}\phi_{\theta_C}(A_i\otimes C^{-1})^{n_i}=\bigotimes_{j=1}^{t}\phi_{\theta_C}(B_j\otimes C^{-1})^{m_j},$$

which is true by Lemma 1.

3. The Picard group of $U_C(r, d)$

We review now the description of the Picard group of the variety $U_C(r, d)$ (resp. $U_C(r, L)$) which parametrizes the semistable vector bundles of rank r and degree d (resp. determinant $L \in J^d(C)$) on a smooth curve C. The reference is [3].

The smooth locus of $U_C(r, d)$ is the set of points $U_C^s(r, d)$ which correspond to stable vector bundles. Also $\operatorname{codim}_{U_C(r,d)}(U_C(r, d)\setminus U_C^s(r, d)) \geq 2$. The space $U_C(r, d)$ is locally factorial (see [3, Theorem A], and so any line bundle on $U_C^s(r, d)$ can be extended uniquely to a line bundle on $U_C(r, d)$. Similarly, one can see that the space $\mathscr{U}(r, d)$ is locally factorial too. The map $\det: U_C(r, d) \to J^d(C)$ which sends $E \mapsto \det E$ has fiber over the point $[L] \in J^d(C)$ the variety $U_C(r, L)$. We have the following (see [3]):

(1) $\operatorname{Pic} U_C(r, L) = \mathbf{Z}$;

(2) $\operatorname{Pic} U_C(r, d) = \mathbf{Z} \oplus \det^* \operatorname{Pic} J^d(C)$.

A geometric description of the generators is given as follows. For a generic choice of a vector bundle F of rank $\frac{r}{n}$ and degree $\frac{-d+r(g-1)}{n}$ where $n =$ g.c.d.(r, d), the set of points $\{E \in U_C(r, d)$ (resp. $E \in U_C(r, L)$) such that $h^0(E \otimes F) \geq 1\}$ defines a divisor in $U_C(r, d)$ (resp. in $U_C(r, L)$). This has been proven in [5]. Note that F has the minimum possible rank for which there exists a degree such that the Euler characteristic $\chi(E\otimes F) = 0$. We denote the induced line bundle by Θ_F (resp. by $\Theta_{L,F}$). The basic facts about these line bundles are

1. The line bundle $\Theta_{L,F}$ on $U_C(r, L)$ does not depend on the choice of F and is the generator of the $\operatorname{Pic} U_C(r, L) \cong \mathbf{Z}$.

2. The line bundle Θ_F on $U_C(r, d)$ depends only on the determinant of the vector bundle F. Namely, if F, F' are two choices as above, then

we have the relation

$$\Theta_F = \Theta_{F'} \otimes \det^*(\text{“}\det F \otimes \det F'^{-1}\text{”}),$$

where $\det F \otimes \det F'^{-1}$ is an element of $J^0(C)$ which can be considered naturally as an element of $\mathrm{Pic}^0 J^d(C)$ (see Lemma 2).

We construct now “canonical” choices of line bundles on $U_C(r, d)$ as follows. Let m be an integer such that $m\frac{-d+r(g-1)}{n}$ is an integral linear combination of the numbers $-d+g-1$ and $2g-2$, i.e.,

$$\text{(1)} \qquad m\frac{-d+r(g-1)}{n} = \alpha(-d+g-1) + \beta(2g-2).$$

The set of all such m's forms a subgroup of the integers with generator

$$\text{(2)} \qquad k_{r,d} = \frac{\text{g.c.d.}\,(2g-2,\, -d+g-1)}{\text{g.c.d.}\,(2g-2,\, -d+g-1,\, \frac{-d+r(g-1)}{n})}.$$

Given F (resp. F') with rank and degree as above, we choose a line bundle M (resp. M') of degree $-d+g-1$, such that $M^\alpha = \det F^m \otimes K^{-\beta}$ (resp. $M'^\alpha = \det F'^m \otimes K^{-\beta}$). There are finitely many such choices, namely α^{2g}. The claim is that the line bundle

$$\text{(3)} \qquad \Theta_F^m \otimes \det^* \theta_M^{-\alpha}$$

does not depend on the choice of F, M. Indeed, we have that

$$\begin{aligned} \Theta_F^m \otimes \det^* \theta_M^{-\alpha} &= \Theta_{F'}^m \otimes \det^*(\text{“}\det F \otimes \det F'^{-1}\text{”})^m \otimes \det^* \theta_M^{-\alpha} \\ &= \Theta_{F'}^m \otimes \det^*((\text{“}\det F \otimes \det F'^{-1}\text{”})^m \otimes \theta_M^{-\alpha}) \\ &= \Theta_{F'}^m \otimes \det^* \theta_{M'}^{-\alpha}, \end{aligned}$$

where the last equality comes from Lemma 3, using that $M^{-\alpha} \otimes \det F^m \otimes \det F'^{-m} = K^\beta \otimes \det F'^{-m} = M'^{-\alpha}$. The line bundles of the above form as in (3) are the canonical choices of line bundles on $U_C(r, d)$. The description of the Picard group of $\mathscr{U}(r, d)$ is given by the following theorem.

Theorem 1. *The restriction of any line bundle on $\mathscr{U}(r, d)$ to the fibers of the map $q: \mathscr{U}(r, d) \to \mathscr{M}_g^0$ is such a canonical choice as in* (3). *Even more, for any choice of integers m, α, β satisfying relation* (1), *there exists a line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$ which restricts to the above canonical choice $\Theta_F^m \otimes \det^* \theta_M^{-\alpha}$ on the fiber $U_C(r, d)$.*

Remark. As we proved in [6], in the special case of the Jacobians $\mathscr{J}^d \to \mathscr{M}_g^0$, i.e., when $r = 1$, the restriction of a line bundle on the fiber $J^d(C)$ has the form θ_M^α, where $M^\alpha = K^\beta$ for some integers α, β. This corresponds to the above situation when $m = 0$; i.e., the line bundle is trivial on the fibers of the map det: $\mathcal{U}(r, d) \to \mathcal{J}^d$ and so it is the pullback of a line bundle from $\mathcal{J}^d$.

4. The space of extensions

We first recall some things about symmetric products of curves. The main reference is [2]. For d large enough, the dth symmetric product $C^{(d)}$ of a smooth curve C can be considered as a projectivized vector bundle over the Jacobian variety $J^d(C)$ in the following way: By fixing a point q_0 in C, there exists a normalized Poincaré bundle $\mathcal{P}_{q_0}$ on the product $J^d(C) \times C$. This is characterized by the properties: $\mathcal{P}_{q_0}|_{\{L\}\times C} \cong L$ and $\mathcal{P}_{q_0}|_{J^d(C)\times\{q_0\}} = \mathcal{O}$. To construct $\mathcal{P}_{q_0}$, we define the map

$$\phi_{q_0}\colon J^d(C) \times C \to J^{g-1}(C), \quad (L, p) \mapsto L \otimes \mathcal{O}((g-d)q_0 - p).$$

Then

$$\mathcal{P}_{q_0} \stackrel{\text{def}}{=} \phi_{q_0}^* \theta \otimes q^* \mathcal{O}((d-g)q_0) \otimes \nu^* \theta^{-1}_{(-d+g-1)q_0}, \tag{4}$$

where ν and q the projections, $\theta = \theta_{\mathcal{O}}$ ($\mathcal{O}$ the trivial line bundle) and $\theta_{(-d+g-1)q_0} \stackrel{\text{def}}{=} \theta_{\mathcal{O}((-d+g-1)q_0)}$, following the notation of §2. We then have that $C^{(d)} \cong \mathbf{P}(\nu_* \mathcal{P}_{q_0})$, and the fiber of the map $u\colon C^{(d)} \cong \mathbf{P}(\nu_* \mathcal{P}_{q_0}) \to J^d(C)$ over a point $[L] \in J^d(C)$ is the projective space $\mathbf{P}(H^0(C, L))$. Given a point p in C, the set $\{D \in C^{(d)}$ such that $D - p \geq 0\}$ defines a divisor which we denote by X_p. As it turns out the divisor X_{q_0} is a section of the tautological line bundle $\mathcal{O}_{\mathbf{P}(\nu_* \mathcal{P}_{q_0})}(1)$ (see [2, p. 309]).

We denote by x the class in the Neron-Severi group of the divisor X_p. This is independent from the choice of the point p. We also denote by δ the class of the diagonal $\Delta \stackrel{\text{def}}{=} \{D \in C^{(d)},\ D = D_{d-2} + 2p$ for some $D_{d-2} \in C^{(d-2)},\ p \in C\}$ in $C^{(d)}$. The pullback of the class θ of the theta divisor in $J^d(C)$ by the Abel-Jacobi map $u\colon C^{(d)} \to J^d(C)$ is given by the MacDonald's formula

$$u^*\theta = (d + g - 1)x - \frac{\delta}{2},$$

(see [2, Proposition 5.1 in p. 358] or [6, Lemma 4]). If C is a curve with general moduli, then it is known that the Neron-Severi group of the

$J^d(C)$ is generated by the class of the theta divisor. From this, one concludes that the Neron-Severi group of $C^{(d)}$ is generated by the class of the pullback of θ and the above-defined class x (see [2, p. 359]); using the MacDonald's formula the generators can be chosen to be $\frac{\delta}{2}$ and x. According to this, given a line bundle $\mathscr{L}$ on the universal dth symmetric product $\mathscr{C}_g^{(d)}$, its restriction to a fiber $C^{(d)}$ is algebraically equivalent to an integral combination $ax + b\frac{\delta}{2}$. Since the curve C is not rational, the classes x and $\frac{\delta}{2}$ are linearly independent. In our paper [6] we show that the coefficient a has to satisfy

$$(*) \qquad 2g-2|a.$$

In the following we are going to see how the relation $(*)$ imposes conditions to line bundles on $\mathscr{U}(r, d)$. To start with, if D is a stable vector bundle of rank r and degree d, then for d large enough—as we are going to assume from now on—we have an exact sequence (see [9])

$$0 \to \mathscr{O}_C \otimes \mathbf{C}^{r-1} \to E \to L \to 0,$$

where $L = \det E$. The extensions of L by $\mathbf{C}^{r-1}$ are parametrized by the points of $H^1(C, L^{-1} \otimes \mathbf{C}^{r-1})$. Let $\mathbf{P}_L = \mathbf{P}(H^1(C, L^{-1} \otimes \mathbf{C}^{r-1}))$. Take a Poincaré bundle $\mathscr{P}$ on $J^d(C) \times C$ and define

$$\mathbf{P} = \mathbf{P}(R^1\nu_*(\mathscr{P}^{-1} \otimes C^{r-1})) \overset{\text{Serre}}{\cong} \mathbf{P}(\nu_*(\mathscr{P} \otimes q^*K)^\vee \otimes \mathbf{C}^{r-1}),$$

where ν and q are the projections of $J^d(C) \times C$. This is a projectivized vector bundle $v\colon \mathbf{P} \to J^d(C)$. According to [4, Proposition 2, application II], there exist a "universal" vector bundle $\mathbf{E}$ on $\mathbf{P} \times C$ and an exact sequence

$$(5) \qquad 0 \to \mathscr{O}_{\mathbf{P}\times C} \otimes \mathbf{C}^{r-1} \to \mathbf{E} \to p_1^*\mathscr{O}_{\mathbf{P}}(-1) \otimes v^\#\mathscr{P} \to 0,$$

(where $v^\# = (v \times 1)^*$, and p_1 is the projection $\mathbf{P} \times C \to \mathbf{P}$) such that for every point x in $\mathbf{P}$, with $v(x) = [L] \in J^d(C)$, its restriction to $\{x\} \times C$

$$0 \to \mathscr{O}_C \otimes \mathbf{C}^{r-1} \to \mathbf{E}_x \to x \otimes L \to 0$$

corresponds to the inclusion $x \to H^1(C, L^{-1} \otimes \mathbf{C}^{r-1})$. Let $\mathbf{P}^s$ be the open of $\mathbf{P}$ consisting of points x with $\mathbf{E}_x$ stable vector bundle. We denote by f the forgetful morphism $f\colon \mathbf{P}^s \to U_C(r, d)$ and also by f again the rational map $f\colon \mathbf{P} \to U_C(r, d)$. The complement of $\mathbf{P}^s$ in $\mathbf{P}$ is of codimension ≥ 2 and so, any line bundle on $\mathbf{P}^s$ extends uniquely to a line bundle on $\mathbf{P}$. We denote now by $\mathbf{P}^{un}$ the bundle over $\mathscr{M}_g^0$

whose fiber over $[C] \in \mathscr{M}_g^0$ is the above space $\mathbf{P}$. We have the following diagram:

$$\begin{array}{ccc} \mathbf{P}^{un} & \xrightarrow{f} & \mathscr{U}(r, d) \\ {\scriptstyle v}\downarrow & \swarrow & \downarrow \\ \mathscr{J}^d & \longrightarrow & \mathscr{M}_g^0 \end{array}$$

The crucial point here is that the variety $\mathbf{P}^{un}$ is a projective bundle over $\mathscr{J}^d$, with fiber over $[L] \in J^d(C)$ isomorphic to the projective space $\mathbf{P}^{(r-1)(d+g-1)-1}$, *but not in general a projectivized one.* This corresponds to the fact that in general there is no Poincaré bundle on $\mathscr{J}^d \times_{\mathscr{M}_g^0} \mathscr{C}_g$, see application at the end of §5. A way to measuring how far $\mathbf{P}^{un}$ is of being a projectivized vector bundle is to determine the minimum positive number l for which there exists a line bundle on $\mathbf{P}^{un}$ whose restrictions to the fibers of the map f is $\mathscr{O}(l)$, where $\mathscr{O}(1)$ is the hyperplane bundle on $\mathbf{P}^{(r-1)(d+g-1)-1}$. We start with a lemma.

Lemma 4. *Let* $\mathbf{P}_1^{un}$ *be the bundle over* $\mathscr{M}_g^0$ *whose fiber over the point* $[C] \in \mathscr{M}_g^0$ *is* $\mathbf{P}(\nu_*(\mathscr{P} \otimes q^*K)^\vee)$, *where* $\mathscr{P}$ *and* ν, q *are as before. Then* $\mathbf{P}_1^{un}$ *is a projective bundle* $v_1 \colon \mathbf{P}_1^{un} \to \mathscr{J}^d$ *for which the corresponding number* l *(definition as above) is the same as that of the projective bundle* $\mathbf{P}^{un}$.

Proof. Over a point $[L] \in J^d(C)$ the fiber of $\mathbf{P}_1^{un}$ is $\mathbf{P}(H^0(C, L\otimes K)^\vee)$ and the fiber of $\mathbf{P}^{un}$ is $\mathbf{P}(H^0(C, L \otimes K)^\vee \otimes \mathbf{C}^{r-1})$. Over a small analytic neighborhood U of $\mathscr{J}^d$ the bundle $\mathbf{P}_1^{un}$ is projectivization of a vector bundle V_U. Let $\phi_1, \cdots, \phi_{d+g-1}$ be a local frame. If $e_1, \cdots, e_{r-1}$ is a frame for the trivial bundle $\mathbf{C}^{r-1}$ over $\mathscr{J}^d$, then over U the bundle $\mathbf{P}^{un}$ is the projectivization of $V_U \otimes \mathbf{C}^{r-1}$ with a local frame $\phi_i \otimes e_j$, $i = 1, \cdots, d+g-1$ and $j = 1, \cdots, r-1$. Consider the diagonal map $V_U \to V_U \otimes \mathbf{C}^{r-1}$ sending $\sum_i a_i\phi_i \mapsto \sum_{ij} a_i\phi_i \otimes e_j$; this induces a morphism $\beta \colon \mathbf{P}_1^{un} \to \mathbf{P}^{un}$. Consider also the map $V_U \otimes \mathbf{C}^{r-1} \to V_U$ sending $\sum_{ij} b_{ij}\phi_i \otimes e_j \mapsto \sum_i(\sum_j b_{ij})\phi_i$; this induces a rational map $\alpha \colon \mathbf{P}^{un} \to \mathbf{P}_1^{un}$. The locus where this is not defined is of codimension $d+g-1$ (≥ 2) in the fibers of $v \colon \mathbf{P}^{un} \to \mathscr{J}^d$. Note that the monodromy on $\mathscr{M}_g^0$ does not cause any problem in the construction of the maps, since the local construction is invariant under the action. Any line bundle on $\mathbf{P}^{un}$ which restricts to $\mathscr{O}(n)$ on the fibers of v pulls back by β to a line bundle on $\mathbf{P}_1^{un}$ with the same property. Similarly, any line bundle on $\mathbf{P}_1^{un}$ with restriction $\mathscr{O}(n)$ pulls back by α and extends *uniquely* to line bundle on $\mathbf{P}^{un}$ with the same property. This proves the lemma.

We have now the following theorem:

Theorem 2. *For the projective bundle* $v: \mathbf{P}^{un} \to \mathscr{J}^d$ *the minimum number* l *for which there exists a line bundle on* $\mathbf{P}^{un}$ *whose restriction on the fibers of the map* v *is* $\mathscr{O}(l)$ *is given by*

$$l = \text{g.c.d}\,(2g-2\,,\, d+g-1)\,.$$

Proof. By Lemma 4 above, it is enough to prove the result for the bundle $\mathbf{P}_1^{un}$. Consider the maps

$$\mathscr{C}_g^{(d)} \times_{\mathscr{M}_g^0} \mathscr{C}_g \stackrel{\phi=u\times 1}{\longrightarrow} \mathscr{J}^d \times_{\mathscr{M}_g^0} \mathscr{C}_g \stackrel{\psi}{\longrightarrow} \mathbf{P}_1^{un}\,,$$

where u is the Abel-Jacobi map, and ψ is the map which sends $(L\,,\,p) \mapsto \{\sigma \in H^0(C\,,\,L\otimes K) \text{ with } \sigma(p) = 0\}$. On $\mathscr{C}_g^{(d)} \times_{\mathscr{M}_g^0} \mathscr{C}_g$ we have a universal bundle $\mathscr{D}_d$; this is the bundle corresponding to the divisor which is the image of the map $\mathscr{C}_g^{(d-1)} \times_{\mathscr{M}_g^0} \mathscr{C}_g \to \mathscr{C}_g^{(d)} \times_{\mathscr{M}_g^0} \mathscr{C}_g$ sending $(D\,,\,p) \mapsto (D+p\,,\,p)$. Note that $\text{class}\,(\mathscr{D}_d|_{C^{(d)}\times\{p\}}) = x$, where x is the class of the divisor X_p defined at the beginning of the section. Let $\mathscr{L}$ be a line bundle on $\mathbf{P}_1^{un}$ which restricts to $\mathscr{O}(s)$ on the fibers of the map $v_1: \mathbf{P}_1^{un} \to \mathscr{J}^d$, where s is an integer. Consider the line bundle $\psi^*\mathscr{L}$. If q is the projection $\mathscr{J}^d \times_{\mathscr{M}_g^0} \mathscr{C}_g \to \mathscr{C}_g$, and ω is the relative dualizing sheaf of the family $\mathscr{C}_g \to \mathscr{M}_g^0$, then the line bundle $\mathscr{P} = \psi^*\mathscr{L} \otimes q^*\omega^{-s}$ has the property $\mathscr{P}|_{\{L\}\times C} \cong L^{\otimes s}$. Also, the class in the Neron-Severi group of the $\mathscr{P}|_{J^d(C)\times\{p\}}$ is independent of the choice of $p \in C$, equal say to $n\theta$ where n is an integer independent of C and p; this is because the Neron-Severi group of the Jacobian of a curve with general moduli is generated by the class of the theta divisor, and since algebraic equivalence is an open (topological) condition, n will not vary over the irreducible space $\mathscr{C}_g$. Therefore from the above-mentioned MacDonald's formula it follows that $\text{class}\,(\phi^*\mathscr{P}|_{C^{(d)}\times\{p\}}) = n((d+g-1)x - \frac{\delta}{2})$.

For each $D \in \mathscr{C}_g^{(d)}$ over $[C]$, we have $\phi^*\mathscr{P}|_{\{D\}\times C} \cong \mathscr{D}_d^{\otimes s}|_{\{D\}\times C} \cong \mathscr{O}(D)^{\otimes s}$. By the see-saw principle (see [8]), there exists a line bundle $\mathscr{R}$ on $\mathscr{C}_g^{(d)}$ such that $\phi^*\mathscr{P} \cong \mathscr{D}_d^{\otimes s} \otimes \pi_1^*\mathscr{R}$, where π_1 is the projection on $\mathscr{C}_g^{(d)}$. Therefore, the restriction of $\mathscr{R}$ to a fiber $C^{(d)}$ of the map $\mathscr{C}_g^{(d)} \to \mathscr{M}_g^0$ has class $[n(d+g-1)-s]x - n\frac{\delta}{2}$. From our basic relation $(*)$, we conclude that $2g-2$ has to divide the coefficient of x; i.e.,

$$n(d+g-1) - s = k(2g-2)\,,$$

where k is an integer. This implies that g.c.d$(2g-2, d+g-1)$ has to divide the number s. To conclude the proof of the theorem we have to prove that there exists a line bundle on $\mathbf{P}_1^{un}$ whose restrictions on the fibers of v is $\mathscr{O}(\text{g.c.d.}(2g-2, d+g-1))$. In the following section we construct such a line bundle.

5. Construction of line bundles

We construct here two line bundles on $\mathbf{P}^{un}$ whose restrictions to the fibers of the map v are $\mathscr{O}(d+g-1)$ and $\mathscr{O}(2g-2)$ respectively. For this, we first do the construction on $\mathbf{P}_1^{un}$, and then pull back by the map α on $\mathbf{P}^{un}$ (see proof of Lemma 4 for the definition of α).

The first line bundle is the dual of the relative dualizing sheaf ω_{v_1} of the family $v_1 : \mathbf{P}_1^{un} \to \mathscr{J}^d$. Since the fibers are projective spaces of dimension $d+g-2$, the dual of ω_{v_1} restricts to $\mathscr{O}(d+g-1)$ on the fibers.

The construction of the second line bundle is a little more complicated. We start with a definition.

Definition 1. For a fixed curve C we denote by $\mathscr{L}_{q_0}$ the tautological bundle $\mathscr{O}_{\mathbf{P}_1}(1)$ of the projectivized bundle $\mathbf{P}_1 \stackrel{\text{def}}{=} \mathbf{P}(\nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee)$, where $\mathscr{P}_{q_0}$ is the normalized Poincaré bundle at q_0, and ν, q are the two projections.

Choose a divisor $\sum_{i=1}^{2g-2} p_i \in H^0(C, K)$ and consider the line bundle $\mathscr{L}_K \cong \bigotimes_{i=1}^{2g-2} \mathscr{L}_{p_i}$ on $\mathbf{P}_1$. As we shall see later (see Definition 2 in §7), this line bundle does not depend on the choice of the section in $H^0(C, K)$. We are going now to prove that there exists a line bundle $\mathscr{L}_K^{un}$ on $\mathbf{P}_1^{un}$, which restricts to $\mathscr{L}_K$ on the fibers. To do this we use the following lemma (see [6]).

Lemma 5. *Let $\mathscr{C}_g \xrightarrow{\pi} \mathscr{M}_g^0$ denote the universal curve over $\mathscr{M}_g^0$, and ω_π the relative dualizing sheaf of π. Then, there is a nonempty Zariski open subset $\mathscr{U}$ of $\mathscr{M}_g^0$ such that on $\pi^{-1}(\mathscr{U})$ there is a holomorphic section of ω_π.*

Proof. $\pi_*\omega_\pi$ is an algebraic bundle on $\mathscr{M}_g^0$. Therefore by Serre's theorem, it can be trivialized on a Zariski open of $\mathscr{M}_g^0$. This is the set $\mathscr{U}$ we are asking for. A trivial section of the bundle $\pi_*\omega_\pi$ over $\mathscr{U}$ corresponds to a holomorphic section of ω_π over $\pi^{-1}\mathscr{U}$. q.e.d.

Note first that we can choose $\mathscr{U}$ such that the restriction of the map π to the above holomorphic section gives an unramified covering of $\mathscr{U}$ of degree $2g-2$. We can now cover the Zariski open $\mathscr{U}$ by open analytic subsets $\{U_a\}$ such that over each U_a there are $2g-2$ sections s_i^a of the map π. Locally over each U_a we can construct a collection of $2g-2$ different maps

$$\phi_{i,a}\colon \mathscr{J}_a^d \times_{U_a} \mathscr{C}_{g,a} \to \mathscr{J}_a^{g-1}, \qquad (L, p) \mapsto L \otimes \mathscr{O}((g-d)q_0 - p),$$

where the subindex a on the bundles means restriction over U_a. Then we define locally Poincaré bundles

$$\mathscr{P}_{i,a} \stackrel{\text{def}}{=} \phi_{i,a}^*\theta \otimes q^*\mathscr{O}((d-g)s_i^a) \otimes \nu^*\theta^{-1}_{(-d+g-1)s_i^a},$$

where the maps ν, q are the projections of $\mathscr{J}_a^d \times_{U_a} \mathscr{C}_{g,a}$, and $\theta_{(-d+g-1)s_i^a}$ is the divisor on $\mathscr{J}_a^d$ whose restriction to $J^d(C)$ is the divisor $\theta_{(-d+g-1)s_i^a([C])}$ (by s_i^a we denote either the map or the image, whatever makes sense). Using these locally defined Poincaré bundles, the restriction of $\mathbf{P}_1^{un}$ over the set U_a can be considered as a projectivized bundle over $\mathscr{J}_a^d$ in $2g-2$ different ways. We denote by $\mathscr{L}_{i,a}$ the corresponding tautological bundles. For each U_a let $\mathscr{L}_a^{un}$ denote the tensor product of all these bundles, which is a line bundle over $\mathbf{P}_1^{un}|_{U_a}$ whose construction remains invariant under the action of the monodromy group. Also by construction the $\mathscr{L}_a^{un}$'s coincide on the overlaps of the set U_a's, and so they fit together and give rise to a line bundle on $\mathbf{P}_1^{un}|_{\mathscr{U}}$ and by extension to a line bundle $\mathscr{L}_K^{un}$ on $\mathbf{P}_1^{un}$. Note that although we may have several possible extensions, their restrictions to the fibers over $\mathscr{M}_g^0$ coincide, and are the above-defined line bundles $\mathscr{L}_K$.

To construct now a line bundle on $\mathbf{P}^{un}$ which restricts to $\mathscr{O}(l)$ on the fibers of the map $v\colon \mathbf{P}^{un} \to \mathscr{J}^d$, consider integers a, b such that $a(2g-2) - b(d+g-1) = l$. Then, if $\mathscr{R} \cong \mathscr{L}_K^{un\otimes a} \otimes \omega_{v_1}^{\otimes b}$, the bundle we are asking for is $\mathscr{R}_1 \cong \alpha^*\mathscr{R}$.

Application. Consider the group

$$A_d = \{n \in Z \text{ such that } \exists \text{a l.b. } \mathscr{P} \text{ on } \mathscr{J}^d \times_{\mathscr{M}_g^0} \mathscr{C}_g \text{ with } \mathscr{P}|_{\{L\}\times C} = L^{\otimes n}\}.$$

Theorem 2 above implies that the generator of the group A_d is the number $l = \text{g.c.d.}\,(d+g-1, 2g-2)$. Indeed, at first $l \in A_d$. If $\mathscr{R}$ is the above line bundle on $\mathbf{P}_1^{un}$, then as we saw in the proof of the theorem, the line bundle $\mathscr{P} \cong \psi^*\mathscr{R} \otimes q^*\omega_{v_1}^{-l}$ has the property that $\mathscr{P}|_{\{L\}\times C} \cong L^{\otimes l}$. On the other hand, using the map ϕ as in the proof of the theorem we conclude l is the generator of A_d. In particular this implies that there exists a Poincaré bundle on $\mathscr{J}^d \times_{\mathscr{M}_g^0} \mathscr{C}_g$ if and only if $\text{g.c.d}(2g-2, d+g-1) = 1$. The latest has been proven in a different way by Mestrano and Ramanan (see [7]).

6. Imposing conditions

For a fixed curve C, let $f\colon \mathbf{P} \to U_C(r, d)$ be the (rational) map defined in §4. Let Θ_F be the line bundle on $U_C(r, d)$ defined in the same section, where $\operatorname{rk} F = \frac{r}{n}$, $\deg F = \frac{-d+r(g-1)}{n}$, $n = \text{g.c.d}(r, d)$. We recall here from [3] how one calculates the $f^*\Theta_F$; note that since f is not defined in a locus of codim ≥ 2, the pullback of Θ_F is uniquely determined. We have the following diagram:

$$\begin{array}{ccccccc}
C & \xleftarrow{p_2} & \mathbf{P} \times C & \xrightarrow{\quad v\times 1 \quad} & J^d(C) \times C & \xrightarrow{q} & C \\
 & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \nu} & & \\
 & & \mathbf{P} & \xrightarrow{\quad v \quad} & J^d(C) & & \\
 & & & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle \det} & & & \\
 & & & U_C(r, d) & & &
\end{array}$$

Tensoring the exact sequence (5) of §4 by $p_2^* F$ and taking direct images to $\mathbf{P}$ we get the induced long exact sequence

$$\begin{aligned}
0 &\to \mathscr{O}_{\mathbf{P}} \otimes H^0(C, \mathbf{C}^{r-1} \otimes F) \to R^0 p_{1*}(\mathbf{E} \otimes p_2^* F) \\
&\to \mathscr{O}_{\mathbf{P}}(-1) \otimes R^0 p_{1*}(v^{\#}\mathscr{P} \otimes p_2^* F) \to \mathscr{O}_{\mathbf{P}} \otimes H^1(C, \mathbf{C}^{r-1} \otimes F) \\
&\to R^1 p_{1*}(\mathbf{E} \otimes p_2^* F) \to \mathscr{O}_{\mathbf{P}}(-1) \otimes R^1 p_{1*}(v^{\#}\mathscr{P} \otimes p_2^* F) \to 0,
\end{aligned}$$

where $v^{\#} \overset{\text{def}}{=} (v \times 1)^*$. Therefore

$$\det p_{1!}(\mathbf{E} \otimes p_2^* F) = \mathscr{O}_{\mathbf{P}}\left(-(r-1)\frac{d}{n}\right) \otimes \det p_{1!}(v^{\#}\mathscr{P} \otimes p_2^* F),$$

(note $-(r-1)\frac{d}{n} = \chi(\det \mathbf{E}_x \otimes F)$). In [3] the authors prove that $f^*(-\Theta_F) \cong \det p_{1!}(\mathbf{E} \otimes p_2^* F)$ (see proof of Theorem C). Now since $p_{1!}(v^{\#}\mathscr{P} \otimes p_2^* F) \cong v^*(\nu_!(\mathscr{P} \otimes q^* F))$ we have

$$f^*(\Theta_F) \cong \mathscr{O}_{\mathbf{P}}\left((r-1)\frac{d}{n}\right) \otimes v^*(\det \nu_!(\mathscr{P} \otimes q^* F))^{-1}. \tag{6}$$

Also if $f_L\colon \mathbf{P}_L \to u(r, L)$ is the restriction map, then

$$f^*(\Theta_{L,F}) \cong \mathscr{O}_{\mathbf{P}_L}\left((r-1)\frac{d}{n}\right),$$

where $\Theta_{L,F}$ is the generator of $\operatorname{Pic} U_C(r, d)$. Combining this with Theorem 2, we impose now conditions for line bundles on $\mathscr{U}(r, d)$. We start with the diagram

$$\begin{array}{ccc} \mathbf{P}^{un} & \xrightarrow{v} & \mathscr{J}^d \\ & {\searrow}^{f} \quad {\nearrow}_{\det} & \\ & \mathscr{U}(r, d) & \end{array}$$

Consider a line bundle $\mathscr{L}$ on $\mathscr{U}(r, d)$ which restricts to $\Theta_{L,F}^{\otimes k}$ on the fiber $U_C(r, L)$ of the map det. Then, $f^*\mathscr{L}|_{\mathbf{P}_L} \cong \mathscr{O}_{\mathbf{P}_L}(k(r-1)\frac{d}{n})$, and Theorem 2 implies that

$$\text{g.c.d.}\,(2g-2, d+g-1) | k(r-1)\tfrac{d}{n}.$$

The minimum of such number k is

$$\frac{\text{g.c.d.}\,(2g-2, d+g-1)}{\text{g.c.d.}\,(2g-2, d+g-1, (r-1)\frac{d}{n})}.$$

Observe that this minimum is the same as the number $k_{r,d}$ in equality (2) of §3. Assume now that the second part of the Theorem 1 is true; i.e., given integers m, α, β satisfying relation (1) of §3, then there exists a line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$ which restricts to the canonical choices $\Theta_F^m \otimes \det^* \theta_M^{-\alpha}$ on the fiber $U_C(r, d)$ over the point $[C] \in \mathscr{M}_g^0$. Thus the above discussion leads to the proof of the first part of the Theorem 1. Indeed, let $\mathscr{L}$ be any line bundle on $\mathscr{U}(r, d)$. The fiber of the map det over a point $[L] \in \mathscr{J}^d$ is $U_C(r, L)$, which has Picard group $\operatorname{Pic} U_C(r, L) \cong \mathbf{Z}[\Theta_{F,L}]$ (see §3). Restricting $\mathscr{L}$ to $U_C(r, L)$, by the above discussion we conclude that $\mathscr{L}|_{U_C(r,L)} \cong \Theta_{F,L}^{\otimes k_{r,d}s}$, s an integer. Therefore by taking $m = k_{r,d}s$ we can find integers α, β satisfying relation (1). Let $\mathscr{L}_{m,\alpha}$ be the "corresponding" line bundle on $\mathscr{U}(r, d)$. Then $\mathscr{L}_{m,\alpha}|_{U_C(r,L)} \cong \Theta_{F,L}^{\otimes m}$ and so $\mathscr{L}|_{U_C(r,L)} \cong \mathscr{L}_{m,\alpha}|_{U_C(r,L)}$. By the see-saw principle there exists a line bundle $\mathscr{M}$ on $\mathscr{J}^d$ such that $\mathscr{L} \cong \mathscr{L}_{m,\alpha} \otimes \det^* \mathscr{M}$. Now using the remark following Theorem 1, we conclude the proof of the first part of this theorem.

7. The generator line bundles on $\mathscr{U}(r, d)$

In this section we construct the above mentioned line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$ and complete the proof of Theorem 1.

Lemma 6. *Let $\mathscr{P}_{q_0}$ be a normalized Poincaré bundle at the point q_0. If $\nu\colon J^d(C)\times C\to J^d(C)$ is the projection map, then*

$$\det \nu_!\mathscr{P}_{q_0}\cong \theta^{-1}_{(-d+g-1)q_0}.$$

More general, if E is a vector bundle on C of rank r_1 and degree d_1, then

$$\det \nu_!(\mathscr{P}_{q_0}\otimes q^*E)\cong \theta^{-(r_1-1)}_{(-d+g-1)q_0}\otimes \theta^{-1}_{\mathscr{O}((-d-d_1+g-1)q_0)\otimes\det E},$$

where $q\colon J^d(C)\times C\to C$ is the projection map.

Proof. We first claim that $\mathscr{P}_{q_0}|_{J^d(C)\times\{p\}}\cong$ "$\mathscr{O}(q_0-p)$". Indeed, with the notation of the construction of $\mathscr{P}_{q_0}$ (see relation (4)) we have

$$\begin{aligned}&\phi^*\theta|_{J^d(C)\times\{p\}}\\ &\quad\cong\mathscr{O}(\{M\in J^d(C)\text{ such that }h^0(C, M\otimes\mathscr{O}((g-d)q_0-p))\geq 1\})\\ &\quad\cong\theta_{(-d+g)q_0-p},\end{aligned}$$

and so $\mathscr{P}_{q_0}|_{J^d(C)\times\{p\}}\cong\theta_{(-d+g)q_0-p}\otimes\theta_{(-d+g)q_0-p}\otimes\theta^{-1}_{(-d+g-1)q_0}$. Then using Lemma 3, the claim is true.

By the Grothendieck-Riemann-Roch theorem one can show that $\det\nu_!\mathscr{P}_{q_0}$ has class θ^{-1}; see [2, Chapter VIII, §2]. Therefore, there exists a line bundle $L\in J^{-d+g-1}(C)$ with $\det\nu_!\mathscr{P}_{q_0}\cong\theta_L^{-1}$. We want to prove that $L\cong\mathscr{O}((-d+g-1)q_0)$. Fix a generic line bundle D_{d-1} of degree $d-1$ on C, and define the map $\psi_1\colon C\to J^d(C)$ which sends $p\mapsto D_{d-1}\otimes\mathscr{O}(p)$. Now consider the diagram

$$\begin{array}{ccccc} C & \xleftarrow{\pi_2} & C\times C & \xrightarrow{\psi=\psi_1\times 1} & J^d(C)\times C\\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \nu}\\ & & C & \xrightarrow{\psi_1} & J^d(C)\end{array}$$

where π_1, π_2, ν are the projection maps. Note that $\psi^*\mathscr{P}_{q_0}|_{\{p\}\times C}\cong D_{d-1}\otimes\mathscr{O}(p)$ and $\psi^*\mathscr{P}_{q_0}|_{C\times\{p\}}\cong\mathscr{O}(p-q_0)$. The last equality is derived from the above claim and the relation $\psi_1^*\theta_L\cong K\otimes D^{-1}_{d-1}\otimes L^{-1}$. Therefore by the theorem of the cube (see [8]), we have

$$\psi^*\mathscr{P}_{q_0}\cong\pi_1^*\mathscr{O}(-q_0)\otimes\pi_2^*D_{d-1}\otimes\mathscr{O}(\Delta).$$

Consider the exact sequence

$$0\to\mathscr{O}_{C\times C}\to\mathscr{O}_{C\times C}(\Delta)\to\mathscr{O}_\Delta(\Delta)\to 0.$$

Tensoring by $\pi_2^* D_{d-1}$ and taking the induced long exact sequence of the projection π_1, we get $\det \pi_{1!}(\Delta \otimes \pi_2^* D_{d-1}) \cong \det(\mathrm{id}_C)_!(\Delta \otimes D_{d-1}) \cong K^{-1} \otimes D_{d-1}$. Therefore

$$\begin{aligned}\det \pi_{1!} \psi^* \mathscr{P}_{q_0} &\cong \det(\mathscr{O}(q_0)^{-1} \otimes \pi_{1!}(\Delta \otimes \pi_2^* D_{d-1})) \\ &\cong \mathscr{O}((-d+g-1)q_0) \otimes K^{-1} \otimes D_{d-1}.\end{aligned}$$

On the other hand $\psi_1^* \det \nu_! \mathscr{P}_{q_0} \cong \psi_1^* \theta_L^{-1} \cong L \otimes D_{d-1} \otimes K^{-1}$. By the base change property we get that $L \otimes D_{d-1} \otimes K^{-1} \cong \mathscr{O}((-d+g-1)q_0) \otimes K^{-1} \otimes D_{d-1}$ and so $L \cong \mathscr{O}((-d+g-1)q_0)$.

For the second identity, if $r_1 = 1$, i.e., $E = \det E$, then a slight modification of the above calculation gives the result. For general rank r_1, we have in the K-group that $[E] = [\mathbf{C}^{r_1 - 1}] \oplus [\det E]$ which proves the lemma.

Lemma 7. *Let E be a vector bundle on a variety X, and let $E' \cong E \otimes L$ where L is a line bundle. Then*

$$\mathscr{O}_{\mathbf{P}E'}(1) \cong \mathscr{O}_{\mathbf{P}E}(1) \otimes \pi^* L^{-1},$$

where π is the canonical map.

Proof. See [4, Chapter II, Lemma 7.9].

Lemma 8. *We have*

$$\mathscr{L}_{q_0} \otimes \mathscr{L}_{p_0}^{-1} \cong v_1^* \text{“}\mathscr{O}(q_0 - p_0)\text{”},$$

where $\mathscr{L}_{q_0}$, $\mathscr{L}_{p_0}$ as in Definition 1 of §5.

Proof. Indeed, by the construction of the normalized Poincaré bundles, we have that $\mathscr{P}_{q_0} \cong \mathscr{P}_{p_0} \otimes \nu^* \text{“}\mathscr{O}(q_0 - p_0)\text{”}$; see claim at the beginning of the proof of Lemma 6. Therefore $\nu_*(\mathscr{P}_{q_0} \otimes q^* K)^\vee \cong \nu_*(\mathscr{P}_{p_0} \otimes q^* K)^\vee \otimes \text{“}\mathscr{O}(p_0 - q_0)\text{”}$, and Lemma 7 concludes the proof.

Definition 2. If M is a line bundle on C, we define $\mathscr{L}_M \overset{\text{def}}{=} \bigotimes_i \mathscr{L}_{p_i} \bigotimes_j \mathscr{L}_{q_j}^{-1}$ where $\sum_i p_i - \sum_j q_j$ is the divisor of a meromorphic section of M. By the above Lemma 8 the definition does not depend on the choice of the divisor. From the same lemma we easily derive the following properties:

(1) $\mathscr{L}_{M_1 \otimes M_2} \cong \mathscr{L}_{M_1} \otimes \mathscr{L}_{M_2}$ where M_1, M_2 are two line bundles.

(2) $v_1^*(\text{“}L_1 \otimes L_2^{-1}\text{”}) \cong \mathscr{L}_{L_1} \otimes \mathscr{L}_{L_2}^{-1}$, where L_1, L_2 are two line bundles of the same degree.

In the following we often use the notation $+$ and $-$ for the “tensor” and the “dual”.

Lemma 9. *If* $L \in J^{-d+g-1}(C)$, *then*

$$v_1^*\theta_L \cong \mathscr{L}_L - \mathscr{L}_K - \omega_{v_1}.$$

Proof. The bundle $v_1 \colon \mathbf{P}_1 = \mathbf{P}(\nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee) \to J^d(C)$ has Euler sequence

$$0 \to \mathscr{O} \to v_1^*\nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee \otimes \mathscr{O}_{\mathbf{P}_1}(1) \to \Omega_{v_1}^\vee \to 0,$$

where Ω_{v_1} is the sheaf of relative differentials of the map v_1. Therefore

$$\det v_1^*\nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee \cong -\omega_{v_1} + (-d-g+1)\mathscr{L}_{q_0}.$$

Since $\det \nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee \cong \theta_{(-d-g+1)\mathscr{O}(q_0)\otimes K}$ (see Lemma 6), we get

$$v_1^*\theta_{(-d-g+1)\mathscr{O}(q_0)\otimes K} \cong -\omega_{v_1} + (-d-g+1)\mathscr{L}_{q_0}.$$

Using Lemma 3 and Lemma 8 yields

$$\begin{aligned} v_1^*\theta_L &\cong v_1^*\theta_{(-d-g+1)\mathscr{O}(q_0)\otimes K} \otimes v_1^*(\text{“}L - ((-d-g+1)\mathscr{O}(q_0) + K)\text{”}) \\ &\cong -\omega_{v_1} + (-d-g+1)\mathscr{L}_{q_0} + \mathscr{L}_L - (-d-g-1)\mathscr{L}_{q_0} - \mathscr{L}_K \\ &\cong -\omega_{v_1} + \mathscr{L}_L - \mathscr{L}_K. \end{aligned}$$

As we saw in the proof of Lemma 4 we have a map $\alpha \colon \mathbf{P} \to \mathbf{P}_1$. On $\mathbf{P}$ we denote again by $\mathscr{L}_L$ the pullback line bundle $\alpha^*\mathscr{L}_L$, and by ω the pull back $\alpha^*\omega_{v_1}$. Note that if we consider $\mathbf{P}$ as a projectivized bundle with the “use” of the Poincaré bundle $\mathscr{P}_{q_0}$, then $\mathscr{O}_{\mathbf{P}}(1) \cong \mathscr{L}_{q_0}$.

Lemma 10. *For the line bundle* Θ_F *on* $U_C(r, d)$,

$$f^*\Theta_F \cong \mathscr{L}_{\det F} - \frac{r}{n}(\mathscr{L}_K + \omega),$$

where $f \colon \mathbf{P} \to U_C(r, d)$ *is the forgetful (rational) map.*

Proof. By Lemmas 6 and 9 we have

$$\begin{aligned} &\det v^*\nu!(\mathscr{P}_{q_0} \otimes q^*F)^{-1} \\ &\quad\cong v^*(\theta^{(r/n)-1}_{(-d+g-1)q_0} \otimes \theta_{(-d+d/n-(r/n)(g-1)+g-1)\mathscr{O}(q_0)\otimes \det F}) \\ &\quad\cong \left(\frac{r}{n} - 1\right)((-d+g-1)\mathscr{L}_{q_0} - \mathscr{L}_K - \omega) \\ &\qquad + \left(-d + \frac{d}{n} - \frac{r}{n}(g-1) + g - 1\right)\mathscr{L}_{q_0} + \mathscr{L}_{\det F} - \mathscr{L}_K - \omega \\ &\quad\cong -(r-1)\frac{d}{n}\mathscr{L}_{q_0} - \frac{r}{n}(\mathscr{L}_K + \omega) + \mathscr{L}_{\det F}. \end{aligned}$$

Now this proves the lemma since

$$f^*\Theta_F \cong (r-1)\tfrac{d}{n}\mathscr{L}_{q_0} \otimes \det v^* \nu!(\mathscr{P} \otimes q^* F)^{-1}$$

(see relation (6)).

From Lemmas 9 and 10 one concludes easily

Theorem 3. *The pullback by the map f of the canonical choices of line bundles on $U_C(r, d)$ is*

$$f^*(\Theta_F^m \otimes \det^* \theta_M^{-\alpha}) \cong \left(\alpha + \beta - \frac{mr}{n}\right)\mathscr{L}_K + \left(\alpha - \frac{mr}{n}\right)\omega;$$

see relations (1), (3) *for the notation.*

Proof. Recall that $M^\alpha \cong \det F^m \otimes K^{-\beta}$, so that $\alpha\mathscr{L}_M \cong m\mathscr{L}_{\det F} - \beta\mathscr{L}_K$. Thus we have

$$\begin{aligned} f^*(\Theta_F^m \otimes \det^* \theta_M^{-\alpha}) &\cong m\mathscr{L}_{\det F} - \frac{mr}{n}\mathscr{L}_K - \frac{mr}{n}\omega - \alpha\mathscr{L}_M + \alpha\mathscr{L}_K + \alpha\omega \\ &\cong \left(\alpha + \beta - \frac{mr}{n}\right)\mathscr{L}_K + \left(\alpha - \frac{mr}{n}\right)\omega. \quad \text{q.e.d.} \end{aligned}$$

We now prove the existence of the line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$. Let $\mathbf{P}_s^{un}$ denote the subset of $\mathbf{P}^{un}$ corresponding to stable points; the complement is of codimension ≥ 2 in $\mathbf{P}^{un}$. In §5, we saw that there exist on $\mathbf{P}_s^{un}$ globally defining line bundles which restrict to $\mathscr{L}_K$ and ω on the fiber over the point $[C] \in \mathscr{M}_g^0$. Therefore there is a line bundle $\mathscr{F}$ on $\mathbf{P}_s^{un}$ which restricts to $(\alpha+\beta-\frac{mr}{n})\mathscr{L}_K + (\alpha - \frac{mr}{n})\omega$ on the fiber over $[C] \in \mathscr{M}_g^0$. The restriction of this bundle to the fibers of the map $f\colon \mathbf{P}_s^{un} \to \mathscr{U}(r, d)$ is trivial (pullback of a line bundle from $U_C(r, d)$). We give now a see-saw principle argument which implies that the above-defined canonical choices of line bundles on the fibers of the map $q\colon \mathscr{U}(r, d) \to \mathscr{M}_g^0$ are actually restrictions of globally defined line bundles on $\mathscr{U}(r, d)$. We are going to use a resolution of the map $f\colon \mathbf{P}_s^{un} \to \mathscr{U}(r, d)$ constructed in [3]. Following that paper, one can construct over $\mathscr{U}(r, d)$ a bundle $\mathbf{T}$ whose fiber over a point $[E] \in \mathscr{U}(r, d)$ is a bundle over the Grassmannian $\mathbf{Gr}(r-1, H^0(C, E))$ with fiber over $[H] \in \mathbf{Gr}(r-1, H^0(C, E))$ to be $\mathbf{P}(\mathrm{Hom}(\mathbf{C}^{r-1}, H))$; see [3, p. 88]. As it turns out the space $\mathbf{P}_s^{un}$ is included in the space $\mathbf{T}$, and the map $f\colon \mathbf{P}_s^{un} \to \mathscr{U}(r, d)$ is extended to the canonical map of the bundle $f_1\colon \mathbf{T} \to \mathscr{U}(r, d)$. The complement of $\mathbf{P}_s^{un}$ in $\mathbf{T}$ is fiberwise a union of two irreducible divisors. Now having the line bundle $\mathscr{F}$ on $\mathbf{P}^{un}$ which is trivial on the fibers of the map f, one can find an extension $\mathscr{F}_1$ of $\mathscr{F}$ to $\mathbf{T}$, which remains trivial on the fibers of the map f_1: for this, just take any extension of $\mathscr{F}$ and then "correct it" by an appropriate combination of the line bundles defined by the above complement divisors. For the map f_1 we can now apply see-saw principle and so there exists a line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r,d)$ such that $\mathscr{F}_1 \cong f_1^*\mathscr{L}_{m,\alpha}$. Using the fact that the pullback map f^* is one-to-one (see [3]), we get that the restrictions of $\mathscr{L}_{m,\alpha}$ to the fibers of the map $q: \mathscr{U}(r,d) \to \mathscr{M}_g^0$ are the above canonical choices, and this concludes the proof of Theorem 1.

Remark 1. In the case of the Jacobian variety $\mathscr{J}^d$, a canonical choice of a line bundle on the fiber $J^d(C)$ has the form θ_L^α, where $L^\alpha \cong K^\beta$. Working with the symmetric product $C^{(d)} \cong \mathbf{P}(\nu_*\mathscr{P})$—assume that d is large enough—we can prove the analogue of the Lemma 10 and Theorem 3 in this case. The corresponding formulas are

(1) $u^*\theta_L \cong \omega_u - \mathscr{L}_L$,
(2) $u^*\theta_L^\alpha \cong \alpha\omega_u - \beta\mathscr{L}_K$,

where $u: C^{(d)} \to J^d(C)$ is the Abel-Jacobi map, and $\mathscr{L}_L$ is defined in a similar way as above. In the same way as before we can see now that there exists a line bundle $\mathscr{L}_\alpha$ which restricts to the above canonical choices on the fibers. This gives a proof of this fact different from that we gave in [6].

Remark 2. The following is also true. If we have a canonical way of choosing a line bundle on the general fiber of the family $q: \mathscr{U}(r,d) \to \mathscr{M}_g^0$, these choices fit together and give rise to a line bundle on $\mathscr{U}(r,d)$.

Acknowledgment

It was a pleasure for me to spend many hours during this year with Tony Pantev in studying together vector bundles and having numerous conversations about this problem. I am grateful to him for his friendship and help.

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